

An inequality related to uncertainty principle in von Neumann algebras

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Abstract

Recently Kosaki proved in [8] an inequality for matrices that can be seen as a kind of new uncertainty principle. Independently, the same result was proved by Yanagi *et al.* in [13]. The new bound is given in terms of Wigner-Yanase-Dyson informations. Kosaki himself asked if this inequality can be proved in the setting of von Neumann algebras. In this paper we provide a positive answer to that question and moreover we show how the inequality can be generalized to an arbitrary operator monotone function.

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1 Introduction

If A, B are selfadjoint matrices and ρ is a density matrix, define

$$\begin{aligned}\text{Cov}_\rho(A, B) &:= \text{Re}\{\text{Tr}(\rho AB) - \text{Tr}(\rho A) \cdot \text{Tr}(\rho B)\} \\ \text{Var}_\rho(A) &:= \text{Cov}_\rho(A, A).\end{aligned}$$

The uncertainty principle reads as

$$\text{Var}_\rho(A)\text{Var}_\rho(B) \geq \frac{1}{4}|\text{Tr}(\rho[A, B])|^2.$$

This inequality can be refined as

$$\text{Var}_\rho(A)\text{Var}_\rho(B) - \text{Cov}_\rho(A, B)^2 \geq \frac{1}{4}|\text{Tr}(\rho[A, B])|^2,$$

(see [5, 12]). Recently a different uncertainty principle has been found [11, 9, 10, 8, 13]. For $\beta \in (0, 1)$ define β -correlation and β -information as

$$\begin{aligned}\text{Corr}_{\rho, \beta}(A, B) &:= \text{Re}\{\text{Tr}(\rho AB) - \text{Tr}(\rho^\beta A \rho^{1-\beta} B)\} \\ I_{\rho, \beta}(A) &:= \text{Corr}_{\rho, \beta}(A, A) = \text{Tr}(\rho A^2) - \text{Tr}(\rho^\beta A \rho^{1-\beta} A),\end{aligned}$$

where the latter coincides with the Wigner-Yanase-Dyson information. It has been proved that

$$\text{Var}_\rho(A)\text{Var}_\rho(B) - \text{Cov}_\rho(A, B)^2 \geq I_{\rho, \beta}(A)I_{\rho, \beta}(B) - \text{Corr}_{\rho, \beta}(A, B)^2. \quad (1.1)$$

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The quantities involved in the previous inequality make a perfect sense in a von Neumann algebra setting (see for example [7]). In ref. [8] Kosaki asked if the inequality (1.1) is true in this more general setting.

In this paper we provide a positive answer to Kosaki question and moreover we show that, once the inequality is formulated in the context of operator monotone functions, the result can be greatly generalized.

2 Preliminaries

Denote by $M_{n,sa}$ the space of complex self-adjoint $n \times n$ matrices, and recall that a function $f : (0, \infty) \rightarrow \mathbb{R}$ is said *operator monotone* if, for any $n \in \mathbb{N}$, any $A, B \in M_{n,sa}$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. Then, $f : (0, \infty) \rightarrow \mathbb{R}$ is operator monotone *iff* for any $A, B \in \mathcal{B}(\mathcal{H})$ such that $0 \leq A \leq B$, it holds $f(A) \leq f(B)$. An operator monotone function is said *symmetric* if $f(x) := xf(x^{-1})$ and *normalized* if $f(1) = 1$. We denote by \mathfrak{F} the class of positive, symmetric, normalized, operator monotone functions.

Examples of operator monotone functions are the so-called Wigner-Yanase-Dyson functions

$$f_\beta(x) := \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)}, \quad \beta \in (0, 1).$$

Returning to a general $f \in \mathfrak{F}$, we associate to it a function $\tilde{f} \in \mathfrak{F}$ [2] defined by

$$\tilde{f}(x) := \frac{1}{2} \left((x + 1) - (x - 1)^2 \frac{f(0)}{f(x)} \right), \quad x > 0.$$

For example

$$\tilde{f}_\beta(x) = \frac{1}{2} (x^\beta + x^{1-\beta}).$$

Definition 2.1. For $A, B \in M_{n,sa}$, $f \in \mathfrak{F}$, and ρ a faithful density matrix, define f -correlation and f -information as

$$\begin{aligned} \text{Corr}_\rho^f(A, B) &:= \text{Re}\{\text{Tr}(\rho AB) - \text{Tr}(R_\rho \tilde{f}(L_\rho R_\rho^{-1})(A) \cdot B)\}, \\ I_\rho^f(A) &:= \text{Corr}_\rho^f(A, A). \end{aligned}$$

Recall that f -information is also known as metric adjusted skew information (see [4]). The following generalization of inequality (1.1) is proved in [2].

Theorem 2.2.

$$\text{Var}_\rho(A) \text{Var}_\rho(B) - \text{Cov}_\rho(A, B)^2 \geq I_\rho^f(A) I_\rho^f(B) - \text{Corr}_\rho^f(A, B)^2.$$

In the next Section we prove that the above inequality holds true in a general von Neumann algebra, thus answering, in particular, the question raised by Kosaki in [8], and recalled above. A different generalization of Theorem 2.2 has been proved in [3].

3 The main result

Let \mathcal{M} be a von Neumann algebra, and ω a normal faithful state on \mathcal{M} , and denote by \mathcal{H}_ω and ξ_ω the GNS Hilbert space and vector, and by S_ω , J_ω and Δ_ω the modular operators associated to ω .

The proof of the main result is divided in a series of Lemmas. In order to deal with unbounded operators, we introduce some sesquilinear forms on \mathcal{H}_ω , and take [6] as our standard reference.

Definition 3.1. Let $f \in \mathfrak{F}$, and define the following sesquilinear forms

$$\begin{aligned}\mathcal{E}(\xi, \eta) &:= \langle \Delta_\omega^{1/2} \xi, \Delta_\omega^{1/2} \eta \rangle, \\ \mathcal{E}_1(\xi, \eta) &:= \mathcal{E}(\xi, \eta) + \langle \xi, \eta \rangle, \\ \mathcal{F}^f(\xi, \eta) &:= \langle \tilde{f}(\Delta_\omega)^{1/2} \xi, \tilde{f}(\Delta_\omega)^{1/2} \eta \rangle, \\ \mathcal{G}^f(\xi, \eta) &:= \frac{1}{2} \mathcal{E}_1(\xi, \eta) - \mathcal{F}^f(\xi, \eta).\end{aligned}$$

It follows from [6], Example VI.1.13, that \mathcal{E} , \mathcal{E}_1 , \mathcal{F}^f are closed, positive and symmetric sesquilinear forms.

Lemma 3.2. Let $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\{\xi_n\}, \{\eta_n\} \subset \mathcal{D}(\Delta_\omega)$ be such that $\xi_n \rightarrow \xi$, $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \rightarrow 0$, $n \rightarrow \infty$, and analogously for η_n and η . Then

$$\begin{aligned}\mathcal{E}(\xi, \eta) &= \lim_{n \rightarrow \infty} \mathcal{E}(\xi_n, \eta_n) = \lim_{n \rightarrow \infty} \langle \xi_n, \Delta_\omega \eta_n \rangle, \\ \mathcal{F}^f(\xi, \eta) &= \lim_{n \rightarrow \infty} \mathcal{F}^f(\xi_n, \eta_n) = \lim_{n \rightarrow \infty} \langle \xi_n, \tilde{f}(\Delta_\omega) \eta_n \rangle.\end{aligned}$$

Proof. It follows from [6] Theorem VI.2.1 that $\mathcal{D}(\Delta_\omega)$ is a core for $\mathcal{D}(\mathcal{E}) \equiv \mathcal{D}(\Delta_\omega^{1/2})$, so that, from [6] Theorem VI.1.21, for any $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$ there is $\{\xi_n\} \subset \mathcal{D}(\Delta_\omega)$ such that $\xi_n \rightarrow \xi$, and $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \rightarrow 0$, $n \rightarrow \infty$. Then $\mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \rightarrow 0$, $m, n \rightarrow \infty$. Now observe that $0 \leq f(x) \leq \frac{1}{2}(x+1)$, for $x > 0$ [2], so that

$$\begin{aligned}\mathcal{F}^f(\xi_n - \xi_m, \xi_n - \xi_m) &= \langle \tilde{f}(\Delta_\omega)^{1/2}(\xi_n - \xi_m), \tilde{f}(\Delta_\omega)^{1/2}(\xi_n - \xi_m) \rangle \\ &= \langle \xi_n - \xi_m, \tilde{f}(\Delta_\omega)(\xi_n - \xi_m) \rangle \\ &\leq \frac{1}{2} \langle \xi_n - \xi_m, \xi_n - \xi_m \rangle + \frac{1}{2} \langle \xi_n - \xi_m, \Delta_\omega(\xi_n - \xi_m) \rangle \\ &= \frac{1}{2} \|\xi_n - \xi_m\|^2 + \frac{1}{2} \mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \rightarrow 0, \quad m, n \rightarrow \infty.\end{aligned}$$

This implies $\xi \in \mathcal{D}(\mathcal{F}^f)$ and $\mathcal{F}^f(\xi_n - \xi, \xi_n - \xi) \rightarrow 0$, $n \rightarrow \infty$.

Therefore, if $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\{\xi_n\}, \{\eta_n\} \subset \mathcal{D}(\Delta_\omega)$ approximate ξ, η in the above sense, we obtain, from [6] Theorem VI.1.12, that $\mathcal{F}^f(\xi, \eta) = \lim_{n \rightarrow \infty} \mathcal{F}^f(\xi_n, \eta_n)$, and analogously for \mathcal{E} . \square

Lemma 3.3.

(i) $\mathcal{D}(\mathcal{F}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$,

(ii) \mathcal{G}^f is a symmetric sesquilinear form on $\mathcal{D}(\mathcal{G}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$, which is positive on $\mathcal{D}(\Delta_\omega^{1/2})$.

Proof. (i) It follows from the proof of the previous Lemma.

(ii) We only need to prove positivity. To begin with, let $\xi \in \mathcal{D}(\Delta_\omega)$. Then, setting $g(x) := \frac{1}{2}(x+1) - \tilde{f}(x) \geq 0$, for all $x > 0$, we have $\mathcal{G}^f(\xi, \xi) = \frac{1}{2} \mathcal{E}_1(\xi, \xi) - \mathcal{F}^f(\xi, \xi) = \frac{1}{2} \langle \xi, \xi \rangle + \frac{1}{2} \langle \xi, \Delta_\omega \xi \rangle - \langle \xi, \tilde{f}(\Delta_\omega) \xi \rangle = \langle \xi, g(\Delta_\omega) \xi \rangle \geq 0$.

Moreover, if $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$, and $\xi_n \in \mathcal{D}(\Delta_\omega)$ is such that $\xi_n \rightarrow \xi$, and $\mathcal{E}(\xi_n - \xi, \xi_n - \xi) \rightarrow 0$, then, from Lemma 3.2 it follows $\mathcal{G}^f(\xi, \xi) = \lim_{n \rightarrow \infty} \mathcal{G}^f(\xi_n, \xi_n) \geq 0$. \square

We can now introduce the main objects of study. In the sequel, we denote by $T \hat{\in} \mathcal{M}$ the fact that T is a closed, densely defined, linear operator on \mathcal{H}_ω , and is affiliated with \mathcal{M} .

Definition 3.4. For any $A, B \hat{\in} \mathcal{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we set $A_0 := A - \langle \xi_\omega, A \xi_\omega \rangle$, $B_0 := B - \langle \xi_\omega, B \xi_\omega \rangle$, and define the bilinear forms

$$\begin{aligned}\text{Cov}_\omega(A, B) &:= \text{Re} \langle A_0 \xi_\omega, B_0 \xi_\omega \rangle, \\ \text{Var}_\omega(A) &:= \text{Cov}_\omega(A, A), \\ \text{Corr}_\omega^f(A, B) &:= \text{Re} \langle A_0 \xi_\omega, B_0 \xi_\omega \rangle - \text{Re} \langle \tilde{f}(\Delta_\omega)^{1/2} A_0 \xi_\omega, \tilde{f}(\Delta_\omega)^{1/2} B_0 \xi_\omega \rangle, \\ I_\omega^f(A) &:= \text{Corr}_\omega^f(A, A).\end{aligned}$$

Remark 3.5. Observe that in the matrix case $\omega = \text{Tr}(\rho)$, for some density matrix ρ , and $\Delta_\omega = L_\rho R_\rho^{-1}$, so that the previous Definition is a true generalization of covariance and f -correlation in the matrix case.

For the reader's convenience, we prove the following folklore result.

Lemma 3.6. $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T \in \widehat{\mathcal{M}}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

Proof. (1) Let us first prove that $\mathcal{D}(\Delta_\omega^{1/2}) \subset \{T\xi_\omega : T \in \widehat{\mathcal{M}}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$. Indeed, let $\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, and define the linear operator $T_0 : x'\xi_\omega \in \mathcal{M}'\xi_\omega \mapsto x'\eta \in \mathcal{H}_\omega$, which is densely defined, and affiliated with \mathcal{M} . Let us show that is preclosed: indeed, if $x'_n\xi_\omega \rightarrow 0$, and $x'_n\eta \rightarrow \zeta$, then, for any $y' \in \mathcal{M}'$, we get

$$\begin{aligned} \langle \zeta, y'\xi_\omega \rangle &= \lim_{n \rightarrow \infty} \langle x'_n\eta, y'\xi_\omega \rangle = \lim_{n \rightarrow \infty} \langle \eta, x'_n{}^* y'\xi_\omega \rangle = \lim_{n \rightarrow \infty} \langle \eta, S_\omega^*(y'^* x'_n\xi_\omega) \rangle \\ &= \lim_{n \rightarrow \infty} \langle y'^* x'_n\xi_\omega, S_\omega\eta \rangle = \lim_{n \rightarrow \infty} \langle x'_n\xi_\omega, y'S_\omega\eta \rangle = 0, \end{aligned}$$

which shows that T_0 is preclosed. Let $T_\eta := \overline{T_0}$. Then, $T_\eta \in \widehat{\mathcal{M}}$, and $T_\eta\xi_\omega = \eta$. It remains to be proved that $\xi_\omega \in \mathcal{D}(T_\eta^*)$. Since $S_\omega\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, we can also consider $T_{S_\omega\eta}$. Let us show that $T_{S_\omega\eta} \subset T_\eta^*$. Indeed, for any $x', y' \in \mathcal{M}'$, we have

$$\langle T_{S_\omega\eta}x'\xi_\omega, y'\xi_\omega \rangle = \langle x'S_\omega\eta, y'\xi_\omega \rangle = \langle S_\omega\eta, x'^*y'\xi_\omega \rangle = \langle y'^*x'\xi_\omega, \eta \rangle = \langle x'\xi_\omega, y'\eta \rangle = \langle x'\xi_\omega, T_\eta y'\xi_\omega \rangle.$$

Then, $\xi_\omega \in \mathcal{D}(T_{S_\omega\eta}) \subset \mathcal{D}(T_\eta^*)$, which shows that $\mathcal{D}(\Delta_\omega^{1/2}) \subset \{T\xi_\omega : T \in \widehat{\mathcal{M}}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

(2) Let us now prove that $\mathcal{D}(\Delta_\omega^{1/2}) \supset \{T\xi_\omega : T \in \widehat{\mathcal{M}}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$. Indeed, if $T \in \widehat{\mathcal{M}}$ is such that $\xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)$, we can consider its polar decomposition $T = v|T|$, and let $e_n := \chi_{[0, n]}(|T|)$, $T_n := v|T|e_n$, for any $n \in \mathbb{N}$. Since $\xi_\omega \in \mathcal{D}(T)$, we have $T_n\xi_\omega = ve_n|T|\xi_\omega \rightarrow T\xi_\omega$. Moreover, since $\xi_\omega \in \mathcal{D}(T^*)$, we have $T_n^*\xi_\omega = |T|e_n v^*\xi_\omega = e_n T^*\xi_\omega \rightarrow T^*\xi_\omega$. Since S_ω is a closed operator, it follows that $T\xi_\omega \in \mathcal{D}(S_\omega) = \mathcal{D}(\Delta_\omega^{1/2})$ [and $S_\omega T\xi_\omega = T^*\xi_\omega$], which is what we wanted to prove. \square

Lemma 3.7. For any $A, B \in \widehat{\mathcal{M}}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we have

(i) $\text{Cov}_\omega(A, B) = \frac{1}{2} \text{Re } \mathcal{E}_1(A_0\xi_\omega, B_0\xi_\omega)$ is a positive bilinear form,

(ii) $\text{Corr}_\omega^f(A, B) = \text{Re } \mathcal{G}^f(A_0\xi_\omega, B_0\xi_\omega)$ is a positive bilinear form.

Proof. (i) Observe that

$$\begin{aligned} \langle B_0\xi_\omega, A_0\xi_\omega \rangle &= \langle B_0^*\xi_\omega, A_0^*\xi_\omega \rangle = \langle J_\omega \Delta_\omega^{1/2} B_0\xi_\omega, J_\omega \Delta_\omega^{1/2} A_0\xi_\omega \rangle \\ &= \langle \Delta_\omega^{1/2} A_0\xi_\omega, \Delta_\omega^{1/2} B_0\xi_\omega \rangle = \mathcal{E}(A_0\xi_\omega, B_0\xi_\omega). \end{aligned}$$

The thesis follows from this and the fact that $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T \in \widehat{\mathcal{M}}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

(ii) It follows from (i) and Lemma 3.3 (ii). \square

Lemma 3.8. Let $\xi, \eta \in \mathcal{H}_\omega$, $\Delta_\omega = \int_0^\infty t de(t)$, and define, for Ω a Borel subset of $[0, \infty)$, $\mu_{\xi\eta}(\Omega) := \text{Re}\langle \xi, e(\Omega)\eta \rangle$, and

$$\mu := \mu_{\xi\xi} \otimes \mu_{\eta\eta} + \mu_{\eta\eta} \otimes \mu_{\xi\xi} - 2\mu_{\xi\eta} \otimes \mu_{\xi\eta}.$$

Then, μ is a bounded positive Borel measure on $[0, \infty)^2$.

Proof. Let Ω_1, Ω_2 be Borel subsets of $[0, \infty)$, and set $e_j := e(\Omega_j)$, $j = 1, 2$. Observe that $|\text{Re}\langle \xi, e_1\eta \rangle \cdot \text{Re}\langle \xi, e_2\eta \rangle| \leq \|e_1\xi\| \cdot \|e_1\eta\| \cdot \|e_2\xi\| \cdot \|e_2\eta\|$, so that

$$\mu(\Omega_1 \times \Omega_2) \geq \|e_1\xi\|^2 \cdot \|e_2\eta\|^2 + \|e_2\xi\|^2 \cdot \|e_1\eta\|^2 - 2\|e_1\xi\| \cdot \|e_1\eta\| \cdot \|e_2\xi\| \cdot \|e_2\eta\| \geq 0.$$

The thesis follows by standard measure theoretic arguments. \square

Theorem 3.9. For any $A, B \in \widehat{\mathcal{M}}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathfrak{F}$, we have

$$\text{Var}_\omega(A) \text{Var}_\omega(B) - \text{Cov}_\omega(A, B)^2 \geq I_\omega^f(A) I_\omega^f(B) - \text{Corr}_\omega^f(A, B)^2.$$

Proof. Set

$$\begin{aligned}
G(A, B) &:= \text{Var}_\omega(A) \text{Var}_\omega(B) - \text{Cov}_\omega(A, B)^2 - I_\omega^f(A) I_\omega^f(B) + \text{Corr}_\omega^f(A, B)^2 \\
&\stackrel{(a)}{=} \frac{1}{2} \mathcal{E}_1(A_0 \xi_\omega, A_0 \xi_\omega) \cdot \frac{1}{2} \mathcal{E}_1(B_0 \xi_\omega, B_0 \xi_\omega) - \left(\frac{1}{2} \text{Re } \mathcal{E}_1(A_0 \xi_\omega, B_0 \xi_\omega) \right)^2 \\
&\quad - \left(\frac{1}{2} \mathcal{E}_1(A_0 \xi_\omega, A_0 \xi_\omega) - \mathcal{F}^f(A_0 \xi_\omega, A_0 \xi_\omega) \right) \left(\frac{1}{2} \mathcal{E}_1(B_0 \xi_\omega, B_0 \xi_\omega) - \mathcal{F}^f(B_0 \xi_\omega, B_0 \xi_\omega) \right) \\
&\quad + \left(\frac{1}{2} \text{Re } \mathcal{E}_1(A_0 \xi_\omega, B_0 \xi_\omega) - \text{Re } \mathcal{F}^f(A_0 \xi_\omega, B_0 \xi_\omega) \right)^2 \\
&= \frac{1}{2} \mathcal{E}_1(A_0 \xi_\omega, A_0 \xi_\omega) \cdot \mathcal{F}^f(B_0 \xi_\omega, B_0 \xi_\omega) + \frac{1}{2} \mathcal{F}^f(A_0 \xi_\omega, A_0 \xi_\omega) \cdot \mathcal{E}_1(B_0 \xi_\omega, B_0 \xi_\omega) \\
&\quad - \mathcal{F}^f(A_0 \xi_\omega, A_0 \xi_\omega) \cdot \mathcal{F}^f(B_0 \xi_\omega, B_0 \xi_\omega) - \text{Re } \mathcal{E}_1(A_0 \xi_\omega, B_0 \xi_\omega) \cdot \text{Re } \mathcal{F}^f(A_0 \xi_\omega, B_0 \xi_\omega) \\
&\quad + (\text{Re } \mathcal{F}^f(A_0 \xi_\omega, B_0 \xi_\omega))^2,
\end{aligned}$$

where in (a) we have used Lemma 3.7. Let us now introduce the function, for $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$,

$$H(\xi, \eta) := \frac{1}{2} \mathcal{E}_1(\xi, \xi) \cdot \mathcal{F}^f(\eta, \eta) + \frac{1}{2} \mathcal{F}^f(\xi, \xi) \cdot \mathcal{E}_1(\eta, \eta) - \mathcal{F}^f(\xi, \xi) \cdot \mathcal{F}^f(\eta, \eta) - \text{Re } \mathcal{E}_1(\xi, \eta) \cdot \text{Re } \mathcal{F}^f(\xi, \eta) + (\text{Re } \mathcal{F}^f(\xi, \eta))^2,$$

and recall that $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T \in \widehat{\mathcal{M}}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$, so that, if A, B are as in the statement of the Theorem, we obtain $G(A, B) = H(A_0 \xi_\omega, B_0 \xi_\omega)$, and to prove the theorem it suffices to show that $H(\xi, \eta) \geq 0$, for all $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$. Observe that, for $\xi, \eta \in \mathcal{D}(\Delta_\omega)$, we get

$$\begin{aligned}
H(\xi, \eta) &= \frac{1}{2} \langle \xi, (1 + \Delta_\omega) \xi \rangle \cdot \langle \eta, \tilde{f}(\Delta_\omega) \eta \rangle + \frac{1}{2} \langle \eta, (1 + \Delta_\omega) \eta \rangle \cdot \langle \xi, \tilde{f}(\Delta_\omega) \xi \rangle \\
&\quad - \langle \xi, \tilde{f}(\Delta_\omega) \xi \rangle \cdot \langle \eta, \tilde{f}(\Delta_\omega) \eta \rangle - \text{Re} \langle \xi, (1 + \Delta_\omega) \eta \rangle \cdot \text{Re} \langle \xi, \tilde{f}(\Delta_\omega) \eta \rangle + (\text{Re} \langle \xi, \tilde{f}(\Delta_\omega) \eta \rangle)^2 \\
&\stackrel{(b)}{=} \frac{1}{2} \int_0^\infty (s+1) d\mu_{\xi\xi}(s) \int_0^\infty \tilde{f}(t) d\mu_{\eta\eta}(t) + \frac{1}{2} \int_0^\infty \tilde{f}(s) d\mu_{\xi\xi}(s) \int_0^\infty (t+1) d\mu_{\eta\eta}(t) \\
&\quad - \int_0^\infty \tilde{f}(s) d\mu_{\xi\xi}(s) \int_0^\infty \tilde{f}(t) d\mu_{\eta\eta}(t) - \frac{1}{2} \int_0^\infty (s+1) d\mu_{\xi\eta}(s) \int_0^\infty \tilde{f}(t) d\mu_{\xi\eta}(t) \\
&\quad - \frac{1}{2} \int_0^\infty \tilde{f}(s) d\mu_{\xi\eta}(s) \int_0^\infty (t+1) d\mu_{\xi\eta}(t) - \int_0^\infty \tilde{f}(s) d\mu_{\xi\eta}(s) \int_0^\infty \tilde{f}(t) d\mu_{\xi\eta}(t) \\
&\stackrel{(c)}{=} \frac{1}{2} \int_{[0, \infty)^2} ((s+1)\tilde{f}(t) + (t+1)\tilde{f}(s) - 2\tilde{f}(s)\tilde{f}(t)) d\mu_{\xi\xi} \otimes \mu_{\eta\eta}(s, t) \\
&\quad - \frac{1}{2} \int_{[0, \infty)^2} ((s+1)\tilde{f}(t) + (t+1)\tilde{f}(s) - 2\tilde{f}(s)\tilde{f}(t)) d\mu_{\xi\eta} \otimes \mu_{\xi\eta}(s, t) \\
&\stackrel{(d)}{=} \frac{1}{4} \iint_{[0, \infty)^2} ((s+1)\tilde{f}(t) + (t+1)\tilde{f}(s) - 2\tilde{f}(s)\tilde{f}(t)) d\mu(s, t),
\end{aligned}$$

where we used in (b) notation as in Lemma 3.8, in (c) Fubini-Tonelli Theorem, and in (d) the symmetries of the first integrand and notation as in Lemma 3.8.

Since μ is a positive measure, and

$$(s+1)\tilde{f}(t) + (t+1)\tilde{f}(s) - 2\tilde{f}(s)\tilde{f}(t) = (s+1 - \tilde{f}(s))\tilde{f}(t) + (t+1 - \tilde{f}(t))\tilde{f}(s) \geq 0,$$

we obtain $H(\xi, \eta) \geq 0$, for any $\xi, \eta \in \mathcal{D}(\Delta_\omega)$.

It follows from Lemma 3.2 that, for any $\xi, \eta \in \mathcal{D}(\Delta_\omega^{1/2})$, we have $H(\xi, \eta) = \lim_{n \rightarrow \infty} H(\xi_n, \eta_n) \geq 0$, which ends the proof. \square

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